

STABILITY OF EXPONENTIAL UTILITY MAXIMIZATION WITH RESPECT TO MARKET PERTURBATIONS

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ABSTRACT. We investigate the continuity of expected exponential utility maximization with respect to perturbation of the Sharpe ratio of markets. By focusing only on continuity, we impose weaker regularity conditions than those found in the literature. Specifically, for markets of the form $S = M + \int \lambda d\langle M \rangle$, we require a uniform bound on the norm of $\lambda \cdot M$ in a suitable *bmo* space.

1. INTRODUCTION

In this paper we provide stability results for the problem of maximizing expected exponential utility. We give conditions under which convergence of markets implies the convergence of optimal terminal wealths as well as their expected utility. Specifically, for markets of the form $S = M + \int \lambda d\langle M \rangle$, our regularity condition consists of a uniform bound on the *bmo*₂ norm of $\lambda \cdot M$. This type of hypothesis is a natural one in mathematical finance and has, for example, appeared in [2] and [8], where it was used in connection with establishing closedness properties of the space of attainable terminal wealths. In the current setting, the *bmo* hypothesis allows us to uniformly approximate the utility of the optimal wealth process, in general unbounded from below, with the utility of wealth processes which *are* bounded from below. With this approximation result in hand, we may use the stability results of [14] for utility functions on $(0, \infty)$ to obtain our main convergence theorem.

In comparison with [14], dealing with the stability problem for utility functions on $(0, \infty)$, our regularity assumption is stronger than the notion of *V*-compactness. It is unknown if that condition suffices in the case of utility functions defined on \mathbb{R} , although consideration of complete markets (see Corollary A.3 in the appendix) suggests that it may be necessary. However, Remark 4.8 indicates that the *bmo* condition we impose may not be a very strong condition. In particular, this remark tells us that for the stability result to hold it is necessary and sufficient to be able to uniformly approximate the utility of the optimal wealth process, and the *bmo* hypothesis lets us do just that.

On the other hand, in comparison with two other extant stability results in the literature, we see that our regularity hypothesis is weaker than in either of those papers, although they provide additional convergence results that are beyond the scope of this article. In [6], the stability of quadratic BSDE's is studied with respect to, among other things, perturbation of the driver. From the natural connection between this class of BSDE's and exponential utility maximization (see [15]),

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the results from [6] allow one to recover stability results about exponential utility maximization, but only under the more restriction assumption of a uniform bound on $\lambda \cdot M$ in the Hardy Space H^∞ , i.e. $\|\lambda \cdot M\|_{H^\infty} \triangleq \|\lambda^2 \cdot \langle M \rangle_T\|_{L^\infty}$. Additionally, it is assumed that the filtration is continuous.

In [18], very strong convergence results are obtained in a narrow class of utility maximization problems, with equilibrium problems in mind. In order to use PDE methods, the setting is exclusively Markov, and the assumptions on market convergence are quite stringent: given a sequence $(\lambda^n)_{n=1, \dots, \infty}$ of drift parameters, essentially $\lambda^n(t) \rightarrow \lambda^\infty(t)$ in $L^\infty([0, T])$. These strong hypotheses are necessary to deduce quantitative estimates about the stability of exponential utility maximization.

Here, we take a different approach. We consider simply the continuity of exponential utility maximization in a general filtration, and are interested in finding minimal regularity conditions under which continuity will hold true. The outline of the paper is as follows. In Section 2, we provide the necessary background definitions to state our main result. In Section 3, we present some preliminaries on the theory of bmo martingales. In Section 4, we apply this theory to give a proof of the main results. Finally, we conclude and give a short appendix of auxiliary results obtainable from V -compactness alone.

2. SETUP AND MAIN RESULT

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0, T]})$ be a filtered probability space satisfying the usual conditions. We assume that $\mathcal{F}_T = \mathcal{F}$. Let M be a continuous local martingale, and let

$$\Lambda \triangleq \left\{ \lambda : \lambda \text{ is a predictable process satisfying } \int_0^T \lambda_u^2 d\langle M \rangle_u < \infty \right\}.$$

For $\lambda \in \Lambda$, define

$$(2.1) \quad S_t^\lambda \triangleq M_t + \int_0^t \lambda_u d\langle M \rangle_u,$$

where $\langle M \rangle = (\langle M \rangle_t)_{t \in [0, T]}$ denotes the quadratic variation of the local martingale M . Along with a numeraire bond, identically equal to 1, each S^λ defines a stock market, in which S^λ is interpreted as the discounted price of a tradeable asset. We assume that every market under consideration satisfies NFLVR, which is equivalent to the existence of an equivalent local martingale measure. It is proven in [4] that all continuous market models satisfying NFLVR have the specific form of (2.1).

We let $S^n \triangleq M + \int \lambda^n d\langle M \rangle$, $n = 1, \dots, \infty$, describe a sequence of markets, and $Z^n \triangleq \mathcal{E}(-\lambda^n \cdot M) = \exp\left(-\int_0^\cdot \lambda^n dM - \frac{1}{2} \int_0^\cdot (\lambda^n)^2 d\langle M \rangle\right)$ is the n th minimal martingale measure. In the exponential utility maximization problem, an agent with utility function $U(x) \triangleq -\exp(-x)$ seeks to maximize $E[U(x + X_T)]$ over a set of admissible wealth processes X that start from initial capital zero. We set $V(y) \triangleq y \log y - y$ for $y > 0$, so that V is the convex dual of U .

Defining the right notion of admissibility is more complicated when U is finite-valued over the whole real line. We state here the most common definition of admissibility at this level of generality, for which we refer to [17]. Let \mathcal{M}^n be the set of equivalent local martingale measures for S^n .

Definition 2.1. For any n , let H be predictable and S^n -integrable. We say that $H \cdot S^n \in \mathcal{A}^n$ if $H \cdot S^n$ is a \mathbb{Q} -martingale for every $\mathbb{Q} \in \mathcal{M}^n$ with finite entropy, that is, $E \left[V \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] < \infty$.

The primal value function u^n , $n = 1, \dots, \infty$, is defined as

$$u^n(x) \triangleq \sup_{X \in \mathcal{A}^n} E[U(x + X_T)], x \in \mathbb{R}.$$

In the stability problem for utility maximization, we seek assumptions on the processes Z^n that ensure the convergence of $u^n(\cdot)$ towards $u^\infty(\cdot)$. Finally, we define the various spaces of *bmo* martingales:

Definition 2.2. Let $1 < p < \infty$. A not necessarily continuous martingale R is in bmo_p , with $\|R\|_{bmo_p} = r$, if there is a minimal constant r such that

$$E[|R_T - R_{\tau-}|^p \mid \mathcal{F}_\tau]^{\frac{1}{p}} \leq r,$$

for all stopping times τ taking values in $[0, T]$.

For $p = 2$, if $\|R\|_{bmo_2} < \infty$, then $\|R\|_{bmo_2} = \text{ess sup}_\tau E[\langle R \rangle_T - \langle R \rangle_{\tau-} \mid \mathcal{F}_\tau]^{\frac{1}{2}}$ also. The equivalence of this representation is easily seen from considering the martingale $M^2 - \langle M \rangle$.

Now we state our main theorems:

Theorem 2.3. Suppose that $Z_T^n \rightarrow Z_T^\infty$ in probability and that $\sup_n \|\lambda^n \cdot M\|_{bmo_2} < \infty$. Then $u^n(\cdot) \rightarrow u^\infty(\cdot)$ pointwise, hence locally uniformly.

Theorem 2.4. Suppose that $Z_T^n \rightarrow Z_T^\infty$ in probability and that $\sup_n \|\lambda^n \cdot M\|_{bmo_2} < \infty$. Then for all x the optimal terminal wealths $\hat{X}_T^n(x)$ converge to $\hat{X}_T^\infty(x)$ in probability as $n \rightarrow \infty$.

3. BMO PRELIMINARIES

Definition 3.1. A positive martingale Y satisfies the Reverse Hölder Inequality $\mathcal{R}_p(\mathbb{P})$ for $p > 1$ with constant K_p and with respect to the measure \mathbb{P} if there exists minimal K_p such that

$$E^\mathbb{P} \left[\frac{Y_T^p}{Y_\tau^p} \mid \mathcal{F}_\tau \right] \leq K_p$$

for all stopping times τ in $[0, T]$.

The following lemma is found in the appendix of [9], and originally in Propositions 5 and 6 of [5].

Lemma 3.2. Suppose that the collection $(\lambda^n \cdot M)_{n \geq 1}$ is bounded in the bmo_2 norm. Then for some $p > 1$ which depends only on this uniform bound, the collection $(Z^n = \mathcal{E}(\lambda^n \cdot M))_{n \geq 1}$ satisfies $\mathcal{R}_p(\mathbb{P})$ with, respectively, uniformly bounded constants C_p^n .

Definition 3.3. A positive martingale Y satisfies $\mathcal{R}_{LL\log L}$ with constant $K_{LL\log L}$ if there exists minimal $K_{LL\log L}$ such that

$$E \left[V \left(\frac{Y_T}{Y_\tau} \right) \mid \mathcal{F}_\tau \right] \leq K_{LL\log L}$$

for all stopping times τ in $[0, T]$.

Definition 3.4. A positive càdlàg process Y satisfies condition (S) if there exist constants $0 < c \leq 1 \leq C$ such that $cY_- \leq Y \leq CY_-$.

The following proposition is mostly in the literature:

Proposition 3.5. Let R be a martingale such that $Y = \mathcal{E}(R)$ is a strictly positive martingale. Then $R \in bmo_2$ and there exists $h > 0$ such that $\Delta R \geq h - 1$ if and only if Y satisfies $\mathcal{R}_{LL\log L}$ and condition (S). The constants $K_{LL\log L}$ and C of Y can be bounded as a function of $\|R\|_{bmo_2}$.

Proof. In the (\Leftarrow) direction, Lemma 2.2 of [9] establishes that $R \in bmo_2$. Now $dY = Y_- dR$ and $\Delta Y = Y_- \Delta R$. By the first inequality of condition (S), $(c - 1)Y_- \leq Y - Y_- = Y_- \Delta R$, implying that $\Delta R \geq c - 1 > -1$.

Now the (\Rightarrow) direction. Since R is in bmo_2 it is locally bounded; indeed, for $n \in \mathbb{N}$, let $\tau_n = \inf\{t : \Delta R_t \geq n\} \wedge T$, and let $r = \|R\|_{bmo_2}$. Then $\|R^{\tau_n}\|_{bmo_2} \leq r$, so that $(\Delta R_{\tau_n})^2 = \Delta \langle R \rangle_{\tau_n} = E[\langle R \rangle_{\tau_n} - \langle R \rangle_{\tau_n-} | \mathcal{F}_{\tau_n}] \leq r$, so that the jumps of R are bounded in magnitude by \sqrt{r} . This implies that R is locally bounded. Then $\Delta Y = Y_- \Delta R \leq \sqrt{r} Y_-$. Hence $Y \leq Y_- + \sqrt{r} Y_-$. Additionally, $\Delta R \geq h - 1$ implies that $Y - Y_- = \Delta Y \geq Y_-(h - 1)$, so $Y \geq h Y_-$. This establishes condition (S). By Lemma 3.2, Y satisfies the reverse Hölder inequality for some $p > 1$. Since $x \log x \leq K' x^p$ for some constant K' , it follows that Y satisfies $\mathcal{R}_{LL\log L}$. Additionally, it is evident that Lemma 3.2 also implies that Y satisfies $\mathcal{R}_{LL\log L}$ with constant $K_{LL\log L}$ only depending on $\|R\|_{bmo_2}$. The arguments in the paragraph above also show that the constant C for Y can also be bounded as a function of $\|R\|_{bmo_2}$. \square

Remark 3.6. For each market n , let \hat{Z}^n be the minimal entropy martingale measure. Let us substantiate the claim that such a \hat{Z}^n exists. Since $\lambda^n \cdot M \in bmo_2$, Theorem 2.3 of [12] implies that Z^n is a true martingale, and Proposition 3.5 implies that Z_T^n has finite entropy. Therefore, the main theorem of [7] implies that the minimal entropy martingale measure exists and is unique.

The next lemma is precisely Lemma 3.1 of [1]. We give a proof for the reader's convenience.

Lemma 3.7. For any n , if Z^n satisfies $\mathcal{R}_{LL\log L}$, with constant K , then \hat{Z}^n satisfies $\mathcal{R}_{LL\log L}$ with a constant less than or equal K .

Proof. By hypothesis, $E\left[\frac{Z_T^n}{Z_\tau^n} \log \frac{Z_T^n}{Z_\tau^n} \middle| \mathcal{F}_\tau\right] \leq K$ for all stopping times τ less than or equal to T . Suppose that \hat{Z}^n does not satisfy $\mathcal{R}_{LL\log L}$ with a constant less than or equal K . Then there exists $\epsilon > 0$, a stopping time σ less than or equal to T , and a set $A \in \mathcal{F}_\sigma$ with $P(A) > 0$ such that

$$E\left[\frac{\hat{Z}_T^n}{\hat{Z}_\sigma^n} \log \frac{\hat{Z}_T^n}{\hat{Z}_\sigma^n} \middle| \mathcal{F}_\sigma\right] \geq K + \epsilon$$

on the set A . Let $\tilde{Z}_t^n \triangleq 1_{\{t < \sigma\}} \hat{Z}_t^n + 1_{\{t \geq \sigma\}} \left(1_A \frac{Z_t^n}{Z_\sigma^n} \hat{Z}_\sigma^n + 1_{A^c} \hat{Z}_t^n\right)$ for $t \in [0, T]$. Then \tilde{Z}^n is the density process of an element of \mathcal{M}^n and satisfies $\tilde{Z}_T^n = 1_A \hat{Z}_\sigma^n \frac{Z_T^n}{Z_\sigma^n} + 1_{A^c} \hat{Z}_T^n$. Thus,

$$\tilde{Z}_T^n \log \tilde{Z}_T^n = 1_{A^c} \hat{Z}_T^n \log \hat{Z}_T^n + 1_A \left(\hat{Z}_\sigma^n \frac{Z_T^n}{Z_\sigma^n} \log \frac{Z_T^n}{Z_\sigma^n} + \frac{Z_T^n}{Z_\sigma^n} \hat{Z}_\sigma^n \log \hat{Z}_\sigma^n \right).$$

Therefore,

$$\begin{aligned}
E \left[\hat{Z}_T^n \log \hat{Z}_T^n | \mathcal{F}_\sigma \right] - E \left[\hat{Z}_T^n \log \hat{Z}_T^n | \mathcal{F}_\sigma \right] &= 1_A \left(\hat{Z}_\sigma^n E \left[\frac{Z_T^n}{Z_\sigma^n} \log \frac{Z_T^n}{Z_\sigma^n} | \mathcal{F}_\sigma \right] + \hat{Z}_\sigma^n \log \hat{Z}_\sigma^n - E \left[\hat{Z}_T^n \log \hat{Z}_T^n | \mathcal{F}_\sigma \right] \right) \\
&= 1_A \hat{Z}_\sigma^n \left(E \left[\frac{Z_T^n}{Z_\sigma^n} \log \frac{Z_T^n}{Z_\sigma^n} | \mathcal{F}_\sigma \right] - E \left[\frac{\hat{Z}_T^n}{\hat{Z}_\sigma^n} \log \frac{\hat{Z}_T^n}{\hat{Z}_\sigma^n} | \mathcal{F}_\sigma \right] \right) \\
&\leq -\epsilon 1_A \hat{Z}_\sigma^n.
\end{aligned}$$

Taking expectations, this contradicts the fact that \hat{Z}^n has minimal entropy. \square

Definition 3.8. *The set $\{Z_T^n : n \in \mathbb{N}\}$ satisfies the V -compactness condition if the set $\{V(Z_T^n) : n \in \mathbb{N}\}$ is uniformly integrable.*

We now show that the bmo_2 hypothesis of (2.3) implies the V -compactness condition, which plays a prominent role in [14]. The next proposition is proven for continuous martingales in [12].

Proposition 3.9. *Suppose $\sup_n \|\lambda^n \cdot M\|_{bmo_2} < \infty$. Then there exists $p > 1$ such that $\sup_n E[(Z_T^n)^p] < \infty$.*

Proof. By the conditional form of Jensen's inequality, the norm $\|\cdot\|_{bmo_1} \leq \|\cdot\|_{bmo_2}$. Let R be an arbitrary element of bmo_2 , and let $n(R) = 2\|R\|_{bmo_1} + \|R\|_{bmo_2}^2$. Without loss of generality, we assume that $\|R\|_{bmo_2} > 0$, and show that the L^p norm of $\mathcal{E}(R)_T$ has an upper bound that only depends on $n(R)$ for some $p > 1$.

Let $\delta = \exp(-pn(R)) < 1$ (so $\log 1/\delta = pn(R)$), and let $\tau = \inf\{t : \mathcal{E}(R)_t > \lambda\}$ for $\lambda > 1$. Arguing in the following paragraph as in [12] (the main difference that we consider time $\tau-$ instead of time τ), we obtain the inequality $P(\mathcal{E}(R)_T / \mathcal{E}(R)_{\tau-} \geq \delta \mid \mathcal{F}_\tau) \geq 1 - \frac{1}{2p}$; indeed, and writing $\mathcal{E}_{ab}(R)$ for $\mathcal{E}(R)_a / \mathcal{E}(R)_b$ for times a, b ,

$$\begin{aligned}
P(\mathcal{E}_{T\tau-}(R) < \delta \mid \mathcal{F}_\tau) &= P(1/\delta < \mathcal{E}_{\tau-T}(R) \mid \mathcal{F}_\tau) \\
&= P\left(pn(R) < R_{\tau-} - R_T + \frac{1}{2}(\langle R \rangle_T - \langle R \rangle_{\tau-}) \mid \mathcal{F}_\tau\right) \\
&\leq \frac{1}{2pn(R)} E[2|R_T - R_{\tau-}| + (\langle R \rangle_T - \langle R \rangle_{\tau-}) \mid \mathcal{F}_\tau] \\
&\leq \frac{n(R)}{2pn(R)} = \frac{1}{2p},
\end{aligned}$$

with the first inequality following from Markov's inequality. This implies that $P(\mathcal{E}_{T\tau-}(R) \geq \delta \mid \mathcal{F}_\tau) \geq 1 - \frac{1}{2p}$.

By Proposition 3.5, $\mathcal{E}(R)$ satisfies the upper bound of condition (S) with a constant C whose size is controlled by $n(R)$, and we have $\mathcal{E}(R)_{\tau-} \geq \frac{1}{C}\mathcal{E}(R)_\tau \geq \frac{1}{C}\lambda$ on $\{\tau < \infty\}$. This yields $P(\mathcal{E}(R)_T \geq \frac{\delta\lambda}{C} \mid \mathcal{F}_\tau) \geq \frac{2p-1}{2p}1_{\{\tau < \infty\}}$. Thus,

$$\begin{aligned}
E[\mathcal{E}(R)_T 1_{\{\mathcal{E}(R)_T \geq \lambda\}}] &\leq E[\mathcal{E}(R)_T 1_{\{\tau < \infty\}}] = E[\mathcal{E}(R)_\tau 1_{\{\tau < \infty\}}] \leq E[C\mathcal{E}(R)_{\tau-} 1_{\{\tau < \infty\}}] \\
&\leq C\lambda P(\tau < \infty) \leq \frac{2C\lambda p}{2p-1} P\left(\mathcal{E}(R)_T \geq \frac{\delta\lambda}{C}\right),
\end{aligned}$$

where the equality above follows from the optional sampling theorem.

Take the inequality $E[\mathcal{E}(R)_T 1_{\{\mathcal{E}(R)_T > \lambda\}}] \leq \frac{2C\lambda p}{2p-1} P(\mathcal{E}(R)_T \geq \frac{\delta\lambda}{C})$, multiply both sides by $(p-1)\lambda^{p-2}$ and integrate with respect to λ from 1 to ∞ :

$$(3.1) \quad \int_1^\infty (p-1)\lambda^{p-2} E[\mathcal{E}(R)_T 1_{\{\mathcal{E}(R)_T > \lambda\}}] d\lambda \leq \int_1^\infty (p-1)\lambda^{p-2} \frac{2C\lambda p}{2p-1} P\left(\mathcal{E}(R)_T \geq \frac{\delta\lambda}{C}\right) d\lambda.$$

Applying Fubini's Theorem to the left hand side of (3.1), we get

$$\begin{aligned} \int_1^\infty (p-1)\lambda^{p-2} E[\mathcal{E}(R)_T 1_{\{\mathcal{E}(R)_T > \lambda\}}] d\lambda &= E\left[\int_1^\infty (p-1)\lambda^{p-2} \mathcal{E}(R)_T 1_{\{\mathcal{E}(R)_T > \lambda\}} d\lambda\right] \\ &= E\left[\mathcal{E}(R)_T \int_1^\infty (p-1)\lambda^{p-2} 1_{\{\mathcal{E}(R)_T > \lambda\}} d\lambda\right] \\ &= E\left[\mathcal{E}(R)_T 1_{\{\mathcal{E}(R)_T > 1\}} \int_1^{\mathcal{E}(R)_T} (p-1)\lambda^{p-2} d\lambda\right] \\ &= E\left[\mathcal{E}(R)_T \left(\mathcal{E}(R)_T^{p-1} - 1\right) 1_{\{\mathcal{E}(R)_T > 1\}}\right]. \end{aligned}$$

After a similar computation for the right hand side, this yields

$$E[(\mathcal{E}(R)_T^p - \mathcal{E}(R)_T) 1_{\{\mathcal{E}(R)_T > 1\}}] \leq \frac{2C(p-1)}{2p-1} E\left[\left(\left(\frac{C}{\delta} \mathcal{E}(R)_T\right)^p - 1\right) 1_{\{\mathcal{E}(R)_T > \frac{\delta}{C}\}}\right].$$

Flipping the position of the second term on each side, we obtain

$$\left(1 - \frac{2C(p-1)}{2p-1} \frac{C^p}{\delta^p}\right) E[\mathcal{E}(R)_T^p 1_{\{\mathcal{E}(R)_T > 1\}}] \leq E[\mathcal{E}(R)_T] - \frac{2C(p-1)}{2p-1} E\left[1_{\{\mathcal{E}(R)_T > \frac{\delta}{C}\}}\right] \leq 1,$$

for any $p > 1$. Hence, by choosing p close enough to 1 so that $\frac{2C(p-1)}{2p-1} \frac{C^p}{\delta^p} < 1$, we establish an upper bound for $E[\mathcal{E}(R)_T^p]$ which depends only on $n(R)$. \square

Corollary 3.10. *Suppose that $\sup_n \|\lambda^n \cdot M\|_{bmo_2} < \infty$. Then $\{V(Z_T^n) : n \in \mathbb{N}\}$ is uniformly integrable.*

Proof. By Proposition 3.9, $\sup_n E[(Z_T^n)^p] < \infty$ for some $p > 1$. As $x^{\tilde{p}}/V(x) \rightarrow \infty$ as $x \rightarrow \infty$, for any $\tilde{p} > 1$, the claim follows from the de la Vallée-Poussin criterion. \square

We make one last digression to the theory of bmo martingales. Specifically, we need the bmo theory of weighted norm inequalities. The following theorem is stated as Theorem 2.16 of [3] without mentioning that the constant C_p in (3.2) can be chosen as the same constant associated with the reverse Hölder inequality. For this fact, see the original proof in Proposition 2 of [5].

Proposition 3.11. *Let $Y = \mathcal{E}(R)$ be a continuous martingale and $\frac{d\mathbb{Q}}{dP} = Y_T$. Then if Y satisfies $\mathcal{R}_p(P)$ with constant C_p , then for each \mathbb{Q} -martingale X and $q = \frac{p}{p-1}$,*

$$(3.2) \quad \lambda^q P(X^* > \lambda) \leq C_p E[|X_T|^q].$$

4. APPROXIMATION OF OPTIMAL WEALTH

In [16], U is approximated by auxiliary utility functions defined on a half axis. For $k \in \mathbb{N}$, we define utility functions $U^{(k)}$ as follows: $U^{(k)} = U$ on $[-k, \infty)$, $U \geq U^{(k)}(x) > -\infty$ for $x > -k - 1$, and $\lim_{x \downarrow -k-1} U^{(k)} = -\infty$. Each $U^{(k)}$ is assumed C^1 , concave, and satisfying the Inada conditions and reasonable asymptotic elasticity. For details on these assumptions, see [16]. $V^{(k)}$ is the convex conjugate of $U^{(k)}$. Since $U^{(k)} \leq U$, $V^{(k)} \leq V$.

For $n = 1, \dots, \infty$, v^n is the dual value function associated to V in market number n :

$$(4.1) \quad v^n(y) \triangleq \inf_{Q \in \mathcal{M}^n} E \left[V \left(y \frac{dQ}{dP} \right) \right], \quad y > 0.$$

For $n = 1, \dots, \infty$ and $k \in \mathbb{N}$, $v^{(n,k)}$ is the value function associated to $V^{(k)}$ in market number n :

$$v^{(n,k)}(y) \triangleq \inf_{Y \in \mathcal{Y}^n} E \left[V^{(k)}(yY_T) \right], \quad y > 0,$$

where \mathcal{Y}^n is the set of supermartingale deflators for S^n :

Definition 4.1. \mathcal{Y}^n is the set of càdlàg processes Y such that $Y_0 = 1$ and $Y(H \cdot S^n)$ is a supermartingale whenever H is predictable, S^n -integrable, such that $H \cdot S^n$ is nonnegative.

Let \mathcal{A}_b^n be the set of wealth processes $H \cdot S^n$ where H is predictable and S -integrable, and $H \cdot S^n$ is bounded from below by a constant. The value functions $u^{(n,k)}$ are defined as follows:

$$u^{(n,k)}(x) \triangleq \sup_{X \in \mathcal{A}_b^n} E \left[U^{(k)}(x + X_T) \right], \quad x > -k - 1.$$

By a shift on the real line (see [16]), one can identify the value functions $v^{(n,k)}, u^{(n,k)}$ with optimization problems that take place over supermartingale deflators and nonnegative wealth processes. We copy verbatim this procedure here.

Let $\tilde{U}^{(k)}(x) \triangleq U^{(k)}(x - (k+1))$, which is finitely valued for $x > 0$. Then $\tilde{U}^{(k)}$ is a utility function of the type encountered in [13], and so there is a unique optimal solution $\bar{X}^{(n,k)}(x) = x + H^{(n,k)} \cdot S^n$ to the optimization problem

$$\tilde{u}^{(n,k)}(x) \triangleq \sup_{X \in \mathcal{A}_b^n} E \left[\tilde{U}^{(k)}(X_T) \right].$$

Then, for $x > -k - 1$, $\hat{X}^{(n,k)} = \bar{X}^{(n,k)}(x + k + 1) - (k + 1)$ is the optimal solution to the optimization problem

$$u^{(n,k)}(x) = \sup_{X \in \mathcal{A}_b^n} E \left[U^{(k)}(x + X_T) \right], \quad x > 0.$$

It follows that $u^{(n,k)}(x) = \tilde{u}^{(n,k)}(x + k + 1)$ for $x > -k - 1$. Let $\tilde{V}^{(k)}$ be the convex conjugate of $\tilde{U}^{(k)}$. Then the convex conjugate $\tilde{v}^{(n,k)}$ of $\tilde{u}^{(n,k)}$ has the form

$$\tilde{v}^{(n,k)}(y) = \inf_{Y \in \mathcal{Y}^n} E \left[\tilde{V}^{(k)}(yY_T) \right] = E \left[\tilde{V}^{(k)} \left(y \tilde{Y}_T^{(n,k)} \right) \right], \quad y > 0.$$

We also have $V^{(k)}(y) = \tilde{V}^{(k)}(y) + (k+1)y$ and $v^{(n,k)}(y) = \tilde{v}^{(n,k)}(y) + (k+1)y$. The main result of [14] implies that for each k , $\lim_{n \rightarrow \infty} \tilde{u}^{(n,k)} = \tilde{u}^{(\infty,k)}$ under the \tilde{V}^k -compactness condition: $\{\tilde{V}^k(Z_T^n) : n \in \mathbb{N}\}$ is uniformly integrable.

Lemma 4.2. *Suppose that $Z_T^n \rightarrow Z_T^\infty$ in probability and $\{Z_T^n : n \in \mathbb{N}\}$ is V -compact. Then for each k , $\lim_{n \rightarrow \infty} u^{(n,k)}(x) = u^{(\infty,k)}(x)$.*

Proof. For each k , $V^{(k)} \leq V$ and $V^{(k)}$ is bounded from below, so $\{V^{(k)}(Z_T^n) : n \in \mathbb{N}\}$ is uniformly integrable. Since $V(x)/x \rightarrow \infty$ as $x \rightarrow \infty$, it is also true that $\{Z_T^n : n \in \mathbb{N}\}$ is uniformly integrable. Given the form of $\tilde{V}^{(k)}$, it now follows that $\{\tilde{V}^{(k)}(Z_T^n) : n \in \mathbb{N}\}$ is uniformly integrable. Hence the main theorem of [14] implies that $\tilde{u}^{(n,k)}(x) \rightarrow \tilde{u}^{(\infty,k)}(x)$. It immediately follows that $u^{(n,k)}(x) \rightarrow u^{(\infty,k)}(x)$. \square

Lemma 4.3. *Suppose that $v^*(y) \triangleq \sup_n v^n(y) < \infty$ for all $y > 0$. Then for all $x \in \mathbb{R}$, $u^*(x) \triangleq \sup_n u^n(x) < 0$.*

Proof. By passing to a subsequence, we can assume that $u^n(0) \rightarrow u^*(0)$. For each n , $u^n(x) = \exp(-x)u^n(0)$, and similarly for $u^*(x)$. Hence, $u^n \rightarrow u^*$ locally uniformly and u^* is concave. Let \bar{v} be the convex dual of u^* . Since v^n and u^n are convex duals, then $\lim_n v^n$ exists and is the convex dual of u^* , and hence is equal to \bar{v} . By definition, $\bar{v} \leq v^*$. Suppose that for some x , $u^*(x) = 0$. Then $u^* \equiv 0$. But, if $u^* \equiv 0$, then it would be that $\bar{v}(y) = \sup_{x \in \mathbb{R}} [u^*(x) - xy] \equiv \infty$, a contradiction. Thus, $u^*(x)$ is bounded away from zero. \square

Let

$$x + \hat{X}^n \triangleq x + \hat{X}^n(0) = \hat{X}^n(x)$$

be the optimal wealth process in market n from initial capital x . This special form for the optimal wealth processes is due to the special form of the exponential utility. Let \mathcal{T} be the set of $[0, T]$ -valued stopping times.

Proposition 4.4. *Suppose that $\sup_n \|\lambda^n \cdot M\|_{bmo_2} < \infty$. Then $\{\exp(-\hat{X}_\tau^n) : n \in \mathbb{N}, \tau \in \mathcal{T}\}$ is uniformly integrable.*

Proof. Recall \hat{Z}^n is the density of the minimal entropy martingale measure for S^n , which we denote by $\hat{\mathbb{Q}}^n$. From Theorem 2.2 of [16], \hat{X}^n is a true $\hat{\mathbb{Q}}^n$ -martingale for each n . From Theorem 2.2 of [16] again, we have $c_n e^{-\hat{X}_T^n} = \hat{Z}_T^n$ for some constant c_n .

Taking conditional expectations under $\hat{\mathbb{Q}}^n$ via Bayes rule, and using the fact that \hat{X}^n is a $\hat{\mathbb{Q}}^n$ -martingale, we obtain

$$\begin{aligned} \log c_n - \hat{X}_\tau^n &= E \left[\frac{\hat{Z}_T^n}{\hat{Z}_\tau^n} \log \hat{Z}_T^n \middle| \mathcal{F}_\tau \right] = E \left[\frac{\hat{Z}_T^n}{\hat{Z}_\tau^n} \left(\log \frac{\hat{Z}_T^n}{\hat{Z}_\tau^n} + \log \hat{Z}_\tau^n \right) \middle| \mathcal{F}_\tau \right] \\ &= E \left[\frac{\hat{Z}_T^n}{\hat{Z}_\tau^n} \log \frac{\hat{Z}_T^n}{\hat{Z}_\tau^n} \middle| \mathcal{F}_\tau \right] + \log \hat{Z}_\tau^n. \end{aligned}$$

Exponentiating the previous inequality, we obtain $\exp(-\hat{X}_\tau^n) = \frac{1}{c_n} \hat{Z}_\tau^n \exp \left(E \left[\frac{\hat{Z}_T^n}{\hat{Z}_\tau^n} \log \frac{\hat{Z}_T^n}{\hat{Z}_\tau^n} \middle| \mathcal{F}_\tau \right] \right) \leq \frac{1}{c_n} e^{\hat{K}_{LL\log L}^n + 1} \hat{Z}_\tau^n$, where $\hat{K}_{LL\log L}^n$ is the $\mathcal{R}_{LL\log L}$ constant of \hat{Z}^n . According to Proposition 3.5 and

Lemma 3.7, $\sup_n \hat{K}_{L\log L}^n < \infty$. By Corollary 3.10, $v^*(y) < \infty$, and so Lemma 4.3 implies that $u^* < 0$. Note that $c_n = -u^n(0)$. Thus, $\inf_n c_n > 0$, so that $\sup_n \frac{1}{c_n} < \infty$. We may then write

$$(4.2) \quad \exp(-\hat{X}_\tau^n) \leq \mathbf{C} \hat{Z}_\tau^n$$

for some constant \mathbf{C} , so that the inequality is valid for all n and all τ . In what follows we will show that the right-hand-side of (4.2) is uniformly integrable, which completes the proof. Since $\sup_n E \left[V(\hat{Z}_T^n) \right] < \infty$ (thanks to V -compactness and Lemma 3.7) and $V(x)/x \rightarrow \infty$ as $x \rightarrow \infty$, the de la Vallée-Poussin criterion implies that $\{\hat{Z}_T^n : n \in \mathbb{N}\}$ is uniformly integrable. Since each \hat{Z}^n is a martingale, this extends to the uniform integrability of $\{\hat{Z}_\tau^n : n \in \mathbb{N}, \tau \in \mathcal{T}\}$. \square

Remark 4.5. *In the literature (see [10]), admissible wealth processes are sometimes defined directly to be those satisfying the conclusion of Proposition 4.4, i.e. having uniformly integrable utility over all stopping times.*

For a càdlàg process Y , let $Y^* \triangleq \sup_{t \in [0, T]} |Y_t| \in \mathcal{F}_T$. For $i \in \mathbb{Z}$, let $\hat{\tau}^{(n, i)} \triangleq \inf\{t : \hat{X}_t^n = i\}$, and let $\hat{X}^{(n, i)} \triangleq (\hat{X}^n)^{\hat{\tau}^{(n, i)}} = \left(\hat{X}_{\hat{\tau}^{(n, i)} \wedge t}^n \right)_{t \in [0, T]}$.

Lemma 4.6. *Suppose that $\sup_n \|\lambda^n \cdot M\|_{bmo_2} < \infty$. Then for each $i \in \mathbb{N}$, the collection $\{(\hat{X}^{(n, i)})^* : n \in \mathbb{N}\}$ is bounded in probability.*

Proof. Let \mathbb{Q}^n be the probability measure associated to the minimal martingale Z^n , which is continuous. By Corollary 3.10, each \mathbb{Q}^n has finite entropy. Theorem 1 of [17] implies that \hat{X}^n is a \mathbb{Q}^n -martingale for each n . Then it is also true that $\hat{X}^{(n, i)}$ is a \mathbb{Q}^n -martingale for each n . Since $\sup_n \|\lambda^n \cdot M\|_{bmo_2} < \infty$, Lemma 3.2 implies that there exists a $p > 1$ such that each Z^n satisfies the Reverse Hölder inequality $\mathcal{R}_p(P)$ with uniformly bounded constant C_p .

By Proposition 3.11, for $q = \frac{p}{p-1}$,

$$\lambda^q P \left((\hat{X}^{(n, i)})^* > \lambda \right) \leq C_p E \left[\left| \hat{X}_T^{(n, i)} \right|^q \right] \leq C_p \left(i^q + \mathbf{C}_i E \left[\exp \left(-\hat{X}_T^{(n, i)} \right) \right] \right) \leq C_p (i^q + \tilde{\mathbf{C}}_i),$$

for constants $\mathbf{C}_i, \tilde{\mathbf{C}}_i$ independent of n , the third inequality a consequence of Proposition 4.4. \square

Proposition 4.7. *Suppose $\sup_n \|\lambda^n \cdot M\|_{bmo_2} < \infty$. Then $u^{(n, k)} \rightarrow u^n$ as $k \rightarrow \infty$, uniformly over the markets n .*

Proof. Let $\epsilon > 0$. Fix $i \in \mathbb{N}$ large enough so that $0 > -\exp(-i) > -\epsilon$. Then

$$(4.3) \quad u^n(0) \geq E \left[U(\hat{X}_T^{(n, i)}) \right] > u^n(0) - \epsilon \text{ for all } n \in \mathbb{N}.$$

For $k \in \mathbb{N}$, let $\hat{X}^{(n, i, -k)} \triangleq (\hat{X}^n)^{\hat{\tau}^{(n, -k)} \wedge \hat{\tau}^{(n, i)}} = (\hat{X}^{(n, i)})^{\hat{\tau}^{(n, -k)}}$. We claim that

$$(4.4) \quad \lim_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} P(\hat{\tau}^{(n, -k)} < \hat{\tau}^{(n, i)}) = 0.$$

Indeed, Lemma 4.6 implies that the collection $\{(\hat{X}^{(n,i)})^* : n \in \mathbb{N}\}$ is bounded in probability. Therefore, $\limsup_{k \rightarrow \infty} \sup_n P((\hat{X}^{(n,i)})^* \geq k) = 0$. But $P(\hat{\tau}^{(n,-k)} < \hat{\tau}^{(n,i)}) \leq P((\hat{X}^{(n,i)})^* \geq k)$, which establishes the claim. We next claim that

$$(4.5) \quad \limsup_{k \rightarrow \infty} \sup_n \left| E \left[U(\hat{X}_T^{(n,i)}) \right] - E \left[U(\hat{X}_T^{(n,i,-k)}) \right] \right| = 0.$$

Let $\epsilon_2 > 0$. Write

$$\begin{aligned} E \left[U(\hat{X}_T^{(n,i,-k)}) \right] &= E \left[U(\hat{X}_T^{(n,i)}) 1_{\{\hat{\tau}^{(n,-k)} \geq \hat{\tau}^{(n,i)}\}} + U(\hat{X}_T^{(n,i,-k)}) 1_{\{\hat{\tau}^{(n,-k)} < \hat{\tau}^{(n,i)}\}} \right] \\ &= E \left[U(\hat{X}_T^{(n,i)}) - U(\hat{X}_T^{(n,i)}) 1_{\{\hat{\tau}^{(n,-k)} < \hat{\tau}^{(n,i)}\}} + U(\hat{X}_T^{(n,i,-k)}) 1_{\{\hat{\tau}^{(n,-k)} < \hat{\tau}^{(n,i)}\}} \right] \end{aligned}$$

According to Proposition 4.4, the set $\{\exp(-\hat{X}_\tau^n) : n \in \mathbb{N}, \tau \in \mathcal{T}\}$ is uniformly integrable, which immediately implies that the set $\{\exp(-\hat{X}_T^{(n,i)}), \exp(-\hat{X}_T^{(n,i,-k)}) : n, k \in \mathbb{N}\}$ is uniformly integrable. Therefore, there exists $\delta = \delta(\epsilon_2) > 0$ such that for any set A , $P(A) < \delta$ implies that $E[U(\hat{X}_T^{(n,i)}) 1_A]$, $E[U(\hat{X}_T^{(n,i,-k)}) 1_A] < \epsilon_2$. According to (4.4), there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ and all $n \in \mathbb{N}$, the sets $\{\hat{\tau}^{(n,-k)} < \hat{\tau}^{(n,i)}\}$ have probability less than δ . Therefore, for $k \geq k_0$ and all $n \in \mathbb{N}$, $\max \left\{ E \left[U(\hat{X}_T^{(n,i)}) 1_{\{\hat{\tau}^{(n,-k)} < \hat{\tau}^{(n,i)}\}} \right], E \left[U(\hat{X}_T^{(n,i,-k)}) 1_{\{\hat{\tau}^{(n,-k)} < \hat{\tau}^{(n,i)}\}} \right] \right\} < \epsilon_2$. Thus, for $k \geq k_0$ and all $n \in \mathbb{N}$, we have

$$\left| E \left[U(\hat{X}_T^{(n,i)}) \right] - E \left[U(\hat{X}_T^{(n,i,-k)}) \right] \right| < 2\epsilon_2,$$

and the claim is established. Then (4.3) and (4.5) imply that

$$(4.6) \quad \limsup_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} \left| u^n(0) - E \left[U(\hat{X}_T^{(n,i,-k)}) \right] \right| \leq \epsilon.$$

Since $\hat{X}^{(n,i,-k)} > -k - 1$, by definition, $u^{(n,k)}(0) \geq \left[U(\hat{X}_T^{(n,i,-k)}) \right]$. Then (4.6) and the fact that $u^{(n,k)} \leq u^n$ imply that for any $\epsilon > 0$,

$$(4.7) \quad \limsup_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} |u^n(0) - u^{(n,k)}(0)| \leq \epsilon,$$

implying that $\limsup_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} |u^{(n,k)}(0) - u^n(0)| = 0$, i.e. that $u^{(n,k)} \rightarrow u^n$ as $k \rightarrow \infty$, uniformly over n . \square

4.1. Proofs of the main theorems.

Proof of Theorem 2.3. It follows from Corollary 3.10 and Lemma 4.2 that for each k , $\lim_{n \rightarrow \infty} u^{(n,k)} = u^{(\infty,k)}$. Proposition 4.7, on the other hand, states that $\lim_{k \rightarrow \infty} u^{(n,k)} = u^n$, uniformly over n . These facts together imply that $\lim_{n \rightarrow \infty} u^n = u^\infty$. \square

Remark 4.8. Since $u^{n,k}$ is increasing in k , a double sequence version of Dini's theorem implies that $u^n \rightarrow u^\infty$ if and only if $u^{n,k} \rightarrow u^n$, uniformly over n . That is, for the stability result in Theorem 2.3 it is necessary and sufficient that one can approximate the value functions u^n uniformly by bounding the wealth processes from below. The bmo hypothesis is what lets us establish this uniform approximation as can be seen from Proposition 4.7.

Proof of Theorem 2.4. As before, for each $k \in \mathbb{N}$, let $\bar{X}_T^{n,k}$ be the optimal terminal wealth in the n th market that satisfies the constraint $\bar{X}_T^{n,k} \geq -k$. By Step 7 in the proof of Theorem 2.2 of [16], we know that as $k \rightarrow \infty$, $U(\bar{X}_T^{n,k}) \rightarrow U(\hat{X}_T^n)$ in L^1 for each $n \in \mathbb{N}$. As a consequence of Proposition 4.7, $E[U(\bar{X}_T^{n,k})] \uparrow E[U(\hat{X}_T^n)]$ as $k \rightarrow \infty$, and the convergence is uniform over n . As $U(\cdot)$ is nonpositive, Scheffe's Lemma then implies that $U(\bar{X}_T^{n,k}) \rightarrow U(\hat{X}_T^n)$ in L^1 as $k \rightarrow \infty$, uniformly over n . L^1 convergence being stronger than L^0 convergence, we also have that $U(\bar{X}_T^{n,k}) \rightarrow U(\hat{X}_T^n)$ in probability, uniformly over all n . Since $\bar{X}_T^{n,k} \rightarrow \bar{X}_T^{\infty,k}$ in probability for all k by Lemma 3.10 of [14], it follows that $U(\hat{X}_T^n) \rightarrow U(\hat{X}_T^\infty)$ in probability.

We claim now that $\{\hat{X}_T^n\}_{n \in \mathbb{N}}$ is bounded in probability. Note that $Z^n \hat{X}_T^n$ is a martingale, so $E[Z_T^n \hat{X}_T^n] = 0$. By Proposition 4.4, $\{U(\hat{X}_T^n)\}_{n \in \mathbb{N}}$ is uniformly integrable, and V -compactness implies that $\{V(Z_T^n)\}_{n \in \mathbb{N}}$ is uniformly integrable. The duality relationship $Z_T^n \hat{X}_T^n \geq U(\hat{X}_T^n) - V(Z_T^n)$ now implies that the negative parts $\{(Z_T^n \hat{X}_T^n)^-\}_{n \in \mathbb{N}}$ are uniformly integrable. Hence $\{Z_T^n \hat{X}_T^n\}_{n \in \mathbb{N}}$ is bounded in L^1 , and also in L^0 . But $Z_T^n \rightarrow Z_T^\infty$ in probability, and Z_T^∞ is strictly positive. Hence $\{Z_T^n\}_{n \in \mathbb{N}}$ is bounded away from zero in probability, and it follows that $\{\hat{X}_T^n\}_{n \in \mathbb{N}}$ is bounded in probability.

Suppose now that \hat{X}_T^n did not converge to \hat{X}_T^∞ in probability. Then there exists an $\epsilon > 0$ such that for infinitely many n , $P(|\hat{X}_T^n - \hat{X}_T^\infty| > \epsilon) > \epsilon$. Now, choose a compact set K such that $P(\hat{X}_T^n \notin K) < \frac{\epsilon}{4}$ for all n . Then $P(|\hat{X}_T^n - \hat{X}_T^\infty| > \epsilon, \text{ and } \hat{X}_T^n, \hat{X}_T^\infty \in K) > \frac{\epsilon}{2}$. For $x, y \in K$, there exists a constant $c > 0$ such that $|U(x) - U(y)| > c|x - y|$. Thus, it follows that for infinitely many n , $P(|U(\hat{X}_T^n) - U(\hat{X}_T^\infty)| > c\epsilon) > \frac{\epsilon}{2}$, contradicting the fact that $U(\hat{X}_T^n) \rightarrow U(\hat{X}_T^\infty)$ in probability. \square

5. CONCLUSION

We have demonstrated that $u^n(x) \rightarrow u^\infty(x)$ when $Z_T^n \rightarrow Z_T^\infty$ in probability together with the condition $\sup_n \|\lambda^n \cdot M\|_{bmo_2} < \infty$. This condition is weaker than any in the literature used to show continuity. In [14] in the setting of utility functions defined on $(0, \infty)$, it is shown that the V -compactness condition is sufficient for attaining continuity, where V is the convex dual of the general utility function U . An obvious question is whether V -compactness is also sufficient for $U(x) = -\exp(-x)$ or general utility functions on \mathbb{R} . The proof from [14] does not generalize, at least in an obvious way, to this setting. Crucial to our analysis was the fact that the bmo -type hypothesis placed structure on the whole time interval $[0, T]$, and not just on the time T -value. The utility maximization problem only explicitly takes place at time T , but for utility functions defined on \mathbb{R} , the subtler definition of admissible strategies pulls intermediate values of wealth processes into the equation.

APPENDIX A.

Here, we record some basic continuity results that are obtainable from the V -compactness hypothesis alone. We note that these do results do not actually depend on the specific structure of the exponential utility, and instead are applicable under the general reasonable asymptotic elasticity assumption, see [16].

Lemma A.1. *Suppose that $Z_T^n \rightarrow Z_T^\infty$ in probability and that $\{Z_T^n : n \in \mathbb{N} \cup \{\infty\}\}$ is V -compact. Then $v^\infty(y) \leq \liminf_{n \rightarrow \infty} v^n(y)$.*

Proof. As in the proof of Lemma 4.2, the V -compactness of $\{Z_T^n : n \in \mathbb{N} \cup \{\infty\}\}$ implies that this set is also $V^{(k)}$ -compact, where $V^{(k)}$ is the dual of the “truncated” utility function $U^{(k)} \leq U$, defined at the beginning of Section 4. By Lemma 4.2 and the main theorem of [14], $\lim_{n \rightarrow \infty} v^{(n,k)}(y) = v^{(\infty,k)}$. By Step 1 of Theorem 2.2 of [16], $v^n(y) = \sup_{k \in \mathbb{N}} v^{(n,k)}(y)$ for each n . Therefore $\liminf_{n \rightarrow \infty} v^n(y) = \liminf_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} v^{(n,k)}(y) \geq \sup_{k \in \mathbb{N}} \liminf_{n \rightarrow \infty} v^{(n,k)}(y) = \sup_{k \in \mathbb{N}} v^{(\infty,k)}(y) = v^\infty(y)$. \square

Lemma A.2. *Suppose that $Z_T^n \rightarrow Z_T^\infty$ in probability and that $\{Z_T^n : n \in \mathbb{N} \cup \{\infty\}\}$ is V -compact. Additionally, suppose that $Z_T^\infty = \hat{Z}_T^\infty$, i.e. the minimal martingale measure and minimal entropy martingale measure coincide. Then $\lim_{n \rightarrow \infty} v^n(y) = v^\infty(y)$. Hence, $\lim_{n \rightarrow \infty} u^n(x) = u^\infty(x)$.*

Proof. Thanks to Lemma A.1, it suffices to show that $\limsup_{n \rightarrow \infty} v^n(y) \leq v^\infty(y)$. By hypothesis, $E[V(yZ_T^n)] \rightarrow E[V(yZ_T^\infty)] = v^\infty(y)$ as $n \rightarrow \infty$. But $v^n(y) \leq \limsup_{n \rightarrow \infty} E[V(yZ_T^n)]$. Therefore, $\limsup_{n \rightarrow \infty} v^n(y) \leq \lim_{n \rightarrow \infty} E[V(yZ_T^n)] = v^\infty(y)$. The last claim follows from the duality between v^n and u^n , see Proposition 3.9 of [14]. \square

Corollary A.3. *Suppose that $Z_T^n \rightarrow Z_T^\infty$ in probability and that the limiting market is complete. Then $\lim_{n \rightarrow \infty} u^n(x) = u^\infty(x)$ if and only if $\{Z_T^n : n \in \mathbb{N}\}$ is V -compact.*

Proof. For the “if” direction, note that in a complete market there is only one equivalent martingale measure, and hence trivially the minimal martingale measure and minimal entropy martingale must agree. Therefore, by Lemma A.2, $\lim_{n \rightarrow \infty} u^n(x) = u^\infty(x)$. The “only if” direction is identical to the proof of Proposition 2.13 of [14]. \square

Remark A.4. *We also note that there are examples of incomplete markets where the minimal martingale and minimal entropy martingale agree. This is the case in a market when one tries to hedge an option written on a non-tradable asset using a geometric Brownian motion correlated with that asset; see e.g. Section 4 of [11].*

Crucial to our proof of Theorem 2.3 was the use of a BMO hypothesis to establish that $u^{(n,k)} \rightarrow u^n$ as $k \rightarrow \infty$, uniformly over all n . One can ask whether this property is superfluous, i.e. if it is possible that $\lim_{n \rightarrow \infty} u^n = u^\infty$ without having $u^{(n,k)} \rightarrow u^n$ as $k \rightarrow \infty$, uniformly over all n . We show below that the answer to this is negative.

Proposition A.5. *$u^n \rightarrow u^\infty$ if and only if $u^{(n,k)} \rightarrow u^{(\infty,k)}$ uniformly over n .*

Proof. The “ \Leftarrow ” implication was the content of Theorem 2.3. For the other direction, let X be the space $\{1, 2, \dots, \infty\}$, with open sets given by $\{n \geq m : n \in \mathbb{N} \cup \{\infty\}\}$ for each $m \in \mathbb{N}$, which is clearly compact.

For each $k \in \mathbb{N}$, the map $n \mapsto u^{(n,k)}(0)$ is continuous by Lemma 4.2. By construction, $u^{(n,k)} \leq u^{(n,k+1)}$ for all n, k , and $u^{(n,k)} \rightarrow u^n$ as $k \rightarrow \infty$ for all n . Therefore, supposing that $n \mapsto u^n(0)$ is continuous, we apply Dini’s Theorem to get the desired result. \square

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